HYPERSONIC VISCOUS FLOW OVER SLENDER BODIES

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In this paper, hypersonic flow of a viscous, heat-conducting gas over a thin body is investigated for the case in which the boundary layer thickness is comparable to, or even greater than, the thickness of the body. In the terminology of [1], this corresponds to the regime of moderate or strong interaction of the inviscid flow with the boundary layer.

From an analysis of the equations of Navier-Stokes, the equations and boundary conditions for the case of three-dimensional flow over a pointed body are developed to the same order of accuracy Δ as the boundary layer equations for plane and axisymmetric flows. Notwithstanding that the relative change of pressure across the region where viscous forces are important is of order Δ, it turns out to be an essential peculiarity of the flow investigated that this transverse change must be taken into account, in contrast to the plane and axisymmetric problems.

Plane and axisymmetric hypersonic flows [2-4] over a slender body may be divided fairly definitely into an inviscid flow and a boundary layer; similarly, a three-dimensional flow can be divided into a viscous and an inviscid flow.

Since in the viscous flow region the pressure depends mainly only on the coordinate x along the flow direction, the inviscid flow will be axisymmetric, to accuracy Δ. The proof is similar to that given in [5]. From the fact that the inviscid flow is nearly axisymmetric, it follows that the ratio of the lift on the body in the case considered to the lift in inviscid flow is equal to zero to order Δ, i.e. to the accuracy being considered.

Also given in the paper is a similarity solution of the system of equations derived for three-dimensional flow; this is a generalization of the well-known exact solution [6] of the equations of the axisymmetric boundary layer with interaction. The asymptotic solution of the equations of self-similar motion near the outer boundary of the region of viscous flow is obtained, which gives the possibility of verifying the correctness of the problem as set up. In conclusion the case of flow over a body of revolution is investigated for angle of attack much smaller than the body thickness (ratio), for which linearization with respect to the axisymmetric flow becomes possible. In Sections 1 to 3 the flow over a pointed body is investigated in detail, and in Section 4 the case of a body with small bluntness is investigated.

1. Hypersonic flow of a viscous, heat-conducting gas over a thin body of arbitrary cross-section (all transverse dimensions much smaller than the length) will be investigated; in particular, bodies of revolution at angle of attack. A cylindrical coordinate system x, r and ω will be used, where the x-axis is parallel to the velocity vector $\mathbf{U}_\infty$ of the undisturbed stream and goes through the foremost point of the body, r is the radius-vector, ω the polar angle, and the coordinates x and r are expressed in units of the body length L. It will be assumed that the equation of the surface of the body has the form $r = \tau R_b(x, \omega)$, where $R_b \sim 1$ is a given function, $\tau \ll 1$ (in the specific case of flow over a thin body of revolution at small angle of attack, the body thickness ratio and the angle of attack have the same order of magnitude, τ).

The following notation is introduced: uU_∞, vU_∞ and wU_∞ are, respectively, the axial (along x), radial (along r) and circumferential components of velocity, $\rho\rho_\infty$ is the density (ρ_∞ is the density of the oncoming flow), $p\rho_\infty U_\infty^2$ is the pressure, HU_∞^2 is the total enthalpy and $\mu\mu_0$ is the coefficient of viscosity (μ_0 is the coefficient of viscosity corresponding to the total temperature of the oncoming flow). It is assumed that the gas is perfect, having a constant value of the adiabatic index κ. The equations of momentum and the equations of continuity, energy and state are written, respectively, in the form

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}+\frac{w}{r}\frac{\partial u}{\partial \omega}\right)+\frac{\partial p}{\partial x}=\frac{4}{3R_0}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{1}{rR_0}\frac{\partial}{\partial r}\left(\mu r\frac{\partial u}{\partial r}\right)+\frac{1}{R_0r^2}\frac{\partial}{\partial \omega}\left(\mu\frac{\partial u}{\partial \omega}\right)-\frac{2}{3R_0}\frac{\partial}{\partial x}\left[\frac{\mu}{r}\frac{\partial (rv)}{\partial r}\right]+\frac{1}{R_0r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial v}{\partial x}\right)-\frac{2}{3R_0r}\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial \omega}\right)+\frac{1}{R_0r}\frac{\partial}{\partial \omega}\left(\mu\frac{\partial w}{\partial x}\right) \qquad (1.1)$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}+\frac{w}{r}\frac{\partial v}{\partial \omega}-\frac{w^2}{r}\right)+\frac{\partial p}{\partial r}=\frac{1}{R_0}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial r}\right)-\frac{2}{3R_0}\frac{\partial}{\partial r}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{1}{R_0}\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{4}{3R_0}\frac{\partial}{\partial r}\left(\mu\frac{\partial v}{\partial r}\right)-\frac{2}{3R_0}\frac{\partial}{\partial r}\left(\frac{\mu v}{r}\right)+\frac{1}{R_0r}\frac{\partial}{\partial \omega}\left(\frac{\mu}{r}\frac{\partial v}{\partial \omega}\right)-\frac{2}{3R_0}\frac{\partial}{\partial r}\left(\frac{\mu}{r}\frac{\partial w}{\partial \omega}\right)+\frac{1}{R_0}\frac{\partial}{\partial \omega}\left[\mu\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right]+\frac{1}{R_0r}\left(2\mu\frac{\partial v}{\partial r}-\frac{2\mu v}{r}-\frac{2\mu}{r}\frac{\partial w}{\partial \omega}\right) \qquad (1.2)$$

$$\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial r}+\frac{w}{r}\frac{\partial w}{\partial \omega}+\frac{vw}{r}\right)+\frac{1}{r}\frac{\partial p}{\partial \omega}=\frac{1}{rR_0}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial w}\right)- \quad (1.3)$$
$$-\frac{2}{3rR_0}\frac{\partial}{\partial \omega}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{1}{R_0}\frac{\partial}{\partial r}\left(\frac{\mu}{r}\frac{\partial v}{\partial \omega}\right)-\frac{2}{3rR_0}\frac{\partial}{\partial \omega}\left(\mu\frac{\partial v}{\partial r}\right)+\frac{1}{R_0}\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right)+$$
$$+\frac{1}{R_0}\frac{\partial}{\partial r}\left[\mu r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right]+\frac{4}{3r^2R_0}\frac{\partial}{\partial \omega}\left(\mu\frac{\partial w}{\partial \omega}\right)+\frac{4}{3r^2R_0}\frac{\partial}{\partial \omega}(\mu v)+\frac{2\mu}{r^2R_0}\frac{\partial v}{\partial \omega}+\frac{2\mu}{R_0}\frac{\partial}{\partial r}\left(\frac{w}{r}\right)$$

$$\frac{\partial(\rho r u)}{\partial x}+\frac{\partial(\rho r v)}{\partial r}+\frac{\partial(\rho w)}{\partial \omega}=0 \quad (1.4)$$

$$\rho u\frac{\partial H}{\partial x}+\rho v\frac{\partial H}{\partial r}+\frac{\rho w}{r}\frac{\partial H}{\partial \omega}=\frac{1}{rR_0}\frac{\partial}{\partial r}\left\{\mu r\frac{\partial}{\partial r}\left[\frac{H}{\sigma}+\left(1-\frac{1}{2\sigma}\right)(u^2+v^2+w^2)\right]\right\}+$$
$$+\frac{1}{r^2R_0}\frac{\partial}{\partial \omega}\left\{\mu\frac{\partial}{\partial \omega}\left[\frac{H}{\sigma}+\left(1-\frac{1}{2\sigma}\right)(u^2+v^2+w^2)\right]\right\}+\frac{1}{R_0}\frac{\partial}{\partial x}\left\{\mu\frac{\partial}{\partial x}\left[\frac{H}{\sigma}+\right.\right.$$
$$\left.\left.+\left(1-\frac{1}{2\sigma}\right)(u^2+v^2+w^2)\right]\right\}-\frac{1}{rR_0}\frac{\partial}{\partial r}\left\{r\mu\left[\left(\frac{\partial rw}{\partial r}-\frac{\partial v}{\partial \omega}\right)\frac{w}{r}-\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial r}\right)u\right]\right\}-$$
$$-\frac{1}{rR_0}\frac{\partial}{\partial \omega}\left\{\mu\left[u\left(\frac{1}{r}\frac{\partial u}{\partial \omega}-\frac{\partial w}{\partial x}\right)-\frac{v}{r}\left(\frac{\partial rw}{\partial r}-\frac{\partial v}{\partial \omega}\right)\right]\right\}-\frac{1}{R_0}\frac{\partial}{\partial x}\left\{\mu\left[v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial r}\right)-\right.\right.$$
$$\left.\left.-w\left(\frac{1}{r}\frac{\partial u}{\partial \omega}-\frac{\partial w}{\partial x}\right)\right]\right\}-\frac{2}{3rR_0}\frac{\partial}{\partial r}\left[\mu r v\left(\frac{1}{r}\frac{\partial rv}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial \omega}+\frac{\partial u}{\partial x}\right)\right]- \quad (1.5)$$
$$-\frac{2}{3rR_0}\frac{\partial}{\partial \omega}\left[\mu w\left(\frac{1}{r}\frac{\partial rv}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial \omega}+\frac{\partial u}{\partial x}\right)\right]-\frac{2}{3R_0}\frac{\partial}{\partial x}\left[\mu u\left(\frac{1}{r}\frac{\partial rv}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial \omega}+\frac{\partial u}{\partial x}\right)\right]$$

$$p=\frac{\varkappa-1}{\varkappa}\rho h,\ \mu=\mu\left(\frac{h}{h_0}\right),\ h=H-\frac{u^2+v^2+w^2}{2},\ h_0=\frac{1}{2},\ R_0=\frac{\rho_\infty U_\infty L}{\mu_0} \quad (1.6)$$

Here, R_0 is the Reynolds number, evaluated at the total temperature of the undisturbed flow.

The equation of the surface which separates the inviscid flow and the boundary layer will be represented in the form $r = \delta R_\delta(x, \omega)$, where $\delta \ll 1$, $R_\delta \sim 1$. It is assumed that the width of the region δ where dissipative processes are important is of the same order* or greater than the body thickness, $\delta \gtrsim \tau$ (with the condition $M_\infty\delta \gtrsim 1$); i.e. the regime is one of moderate or strong interaction of the inviscid flow with the boundary layer [1].

Due to the low density in the region of viscous flow, it may be shown that the inviscid flow is over a body whose surface coincides with R_δ. From the small perturbation theory of hypersonic flow [1], it follows that the thickness of the perturbed region of the inviscid flow

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* For the regime $\delta \sim \tau$, in a specific case of flow over a body of revolution, fulfillment of this condition for zero angle of attack does not ensure that the given theory is valid when there is angle of attack $\alpha \sim \tau$. It is necessary that, for angle of attack, the condition $\delta \sim \tau$ be fulfilled in all meridional planes.

is also of order δ. From this, we find that for the independent variables throughout the perturbed region of the flow: $x \sim 1$, $r \sim \delta$ and $\omega \sim 1$. The velocity component u along the x-axis is everywhere of order unity. Using these estimates and equating, in order of magnitude, the second and third terms of the continuity equation (1.4), we find for the other two velocity components, $v \sim \delta$ and $w \sim \delta$, everywhere in the perturbed region.

From the properties of inviscid, hypersonic flow, we obtain the estimate $p \sim \delta^2$ for the pressure. As is well known [3,4], the enthalpy in the boundary layer at hypersonic speeds, $h \cong H - u^2/2$, is of the same order as the total enthalpy, even with strong cooling, i.e. $h \sim 1/2$ (from this it follows that $\mu \sim 1$). Using these estimates for the pressure and enthalpy, we obtain, from the equation of state (1.6), $\rho \sim \delta^2/\varepsilon$, where $\varepsilon = (\kappa - 1)/2\kappa$. We require that the error of the theory be $\Delta = \delta^2/\varepsilon \ll 1$. Equating, in order of magnitude, one of the convective terms (they are all of the same order) and one of the larger terms in equation (1.1), taking into account the estimates obtained above ($u \sim 1$, $x \sim 1$, $\rho \sim \delta^2/\varepsilon$, $\mu \sim 1$), we have, for the region of viscous flow

$$\rho u \frac{\partial u}{\partial x} \sim \frac{1}{rR_0} \frac{\partial}{\partial r}\left(\mu r \frac{\partial u}{\partial r}\right), \quad \frac{\delta^2}{\varepsilon} \sim \frac{1}{R_0 \delta^2}, \quad \delta \sim \left(\frac{\varepsilon}{R_0}\right)^{1/4} \tag{1.7}$$

From equations (1.2) and (1.3) we find that the components of the pressure gradient in the radial and circumferential directions are of the same order of magnitude. From this is obtained an estimate for the change of pressure Δp between any two points in the plane x = const lying in the region of viscous flow (the distance between these points is evidently of order not greater than δ)

$$\frac{\partial p}{\partial r} \sim \rho u \frac{\partial v}{\partial x} \sim \frac{\delta^3}{\varepsilon}, \quad \frac{1}{r} \frac{\partial p}{\partial \omega} \sim \rho u \frac{\partial w}{\partial x} \sim \frac{\delta^3}{\varepsilon}, \quad \Delta p = \int \left(\frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial \omega} d\omega\right) \sim \frac{\delta^4}{\varepsilon} \tag{1.8}$$

From (1.8) it follows that $\Delta p/p \sim \delta^4/\varepsilon\delta^2 \sim \delta^2/\varepsilon = \Delta$, i.e. within the error Δ, the pressure may be taken to be dependent only on x.

Consequently, the inviscid flow must be axisymmetric to the same accuracy, since in the viscous region there can be no significant changes of pressure in the circumferential direction.

2. In the development of the initial equations and in writing the boundary conditions, we shall not make use of the smallness of the quantity ε. We shall define δ by the equality $\delta = R_0^{-1/4}$. We shall represent the independent variables and the functions to be determined in the viscous flow region in the form

$$x = x_0, \quad r = \delta r_0, \quad \omega = \omega_0, \quad u = u_0(x_0, r_0, \omega_0), \quad v = \delta v_0(x_0, r_0, \omega_0)$$
$$w = \delta w_0(x_0, r_0, \omega_0), \quad p = \delta^2 p_0(x_0) + \delta^4 p_1(x_0, r_0, \omega_0)$$
$$\rho = \delta^2 \rho_0(x_0, r_0, \omega_0), \quad H = H_0(x_0, r_0, \omega_0), \quad \delta = R_0^{-1/4} \tag{2.1}$$

where quantities with subscripts 0 and 1 are of order unity.

The pressure is represented as the sum of two components, each of which enters into the equations, as may be seen from what follows. The outer boundary of the viscous flow region, whose equation is $r = \delta R_\delta(x, \omega)$, is taken as the boundary of the effective body for the inviscid flow. Within an error of order δ^2, the inviscid flow must be axisymmetric. Therefore, the equations of the surface dividing the viscous and inviscid flows may be represented in the form

$$r = \delta R_\delta(x) + \delta^3 R_1(x, \omega) \quad \text{or} \quad r_0 = R_\delta(x_0) + o(\delta^2) \tag{2.2}$$

We shall neglect the bluntness of the effective body (2.2). To the hypersonic flow over the effective body (2.2) with thickness of order δ we may apply the well-known analogy with nonstationary flow [1], which is valid up to error δ^2. Within this error, it is evidently possible to set the quantity $w \sim \delta^3$ in the equations of inviscid flow equal to zero. Using the order of magnitude estimates for inviscid flow over a slender body [1], we have

$$u = 1 + o(\delta^2), \quad v = \delta v_0(x_0, r_0), \quad w = o(\delta^3) \approx 0$$
$$p = \delta^2 p_0(x_0, r_0), \quad \rho = \rho_0(x_0, r_0), \quad H = {}^1/_2 \tag{2.3}$$

where quantities with subscript 0 are of order unity, and the independent variables x, r and ω transform according to equations (2.1). The equation of the shock wave to the accuracy adopted is $r = \delta R_s(x)$ or $r_0 = R_s(x_0)$.

After going over to variables with subscript 0, which will be omitted for brevity, in accordance with equation (2.3), the equations of inviscid axisymmetric flow can be written in the form

$$\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \quad \frac{\partial \rho r}{\partial x} + \frac{\partial \rho r v}{\partial r} = 0, \quad \frac{\partial}{\partial x}\frac{p}{\rho^\varkappa} + v\frac{\partial}{\partial r}\frac{p}{\rho^\varkappa} = 0 \tag{2.4}$$

The boundary conditions at the shock wave $r = R_s(x)$ take the form (a prime denotes differentiation with respect to x)

$$v = \frac{2}{\varkappa + 1} R_s' \left(1 - \frac{1}{M_\infty^2 \delta^2 R_s'^2}\right), \quad p = \frac{2R_s'^2}{\varkappa + 1}\left(1 - \frac{\varkappa - 1}{2\varkappa M_\infty^2 \delta^2 R_s'^2}\right)$$
$$\rho = \frac{(\varkappa + 1)}{(\varkappa - 1)}\left(1 + \frac{2}{\varkappa - 1}\frac{1}{M_\infty^2 \delta^2 R_s'^2}\right) \tag{2.5}$$

while on the surface of the effective body $r = R_\delta(x)$ the tangency condition is applied: $v = R_\delta'(x)$.

Dropping terms in the viscous flow equations (1.1) to (1.6) which are of order δ^2 compared to the remaining ones, we have, in the new variables (the subscript 0 is omitted for brevity)

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}+\frac{w}{r}\frac{\partial u}{\partial \omega}\right)+\frac{\partial p}{\partial x}=\frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \omega}\left(\mu\frac{\partial u}{\partial \omega}\right) \tag{2.6}$$

$$\begin{aligned}\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}+\frac{w}{r}\frac{\partial v}{\partial \omega}-\frac{w^2}{r}\right)+\frac{\partial p_1}{\partial r}&=\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial r}\right)-\frac{2}{3}\frac{\partial}{\partial r}\left(\mu\frac{\partial u}{\partial x}\right)+\\&+\frac{4}{3}\frac{\partial}{\partial r}\left(\mu\frac{\partial v}{\partial r}\right)-\frac{2}{3}\frac{\partial}{\partial r}\left(\frac{\mu v}{r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \omega}\left(\mu\frac{\partial v}{\partial \omega}\right)-\frac{2}{3}\frac{\partial}{\partial r}\left(\frac{\mu}{r}\frac{\partial w}{\partial \omega}\right)+\\&+\frac{\partial}{\partial \omega}\left[\mu\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right]+\frac{1}{r}\left(2\mu\frac{\partial v}{\partial r}-\frac{2\mu v}{r}-\frac{2\mu}{r}\frac{\partial w}{\partial \omega}\right)\end{aligned} \tag{2.7}$$

$$\begin{aligned}\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial r}+\frac{w}{r}\frac{\partial w}{\partial \omega}+\frac{vw}{r}\right)+\frac{1}{r}\frac{\partial p_1}{\partial \omega}&=\frac{1}{r}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial \omega}\right)-\frac{2}{3r}\frac{\partial}{\partial \omega}\left(\mu\frac{\partial u}{\partial x}\right)+\\&+\frac{\partial}{\partial r}\left(\frac{\mu}{r}\frac{\partial v}{\partial w}\right)-\frac{2}{3r}\frac{\partial}{\partial \omega}\left(\mu\frac{\partial v}{\partial r}\right)+\frac{\partial}{\partial r}\left[\mu r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right]+\frac{4}{3r^2}\frac{\partial}{\partial \omega}\left(\mu\frac{\partial w}{\partial \omega}\right)+\\&+\frac{4}{3r^2}\frac{\partial}{\partial \omega}(\mu v)+\frac{2\mu}{r^2}\frac{\partial v}{\partial \omega}+2\mu\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\end{aligned} \tag{2.8}$$

$$\frac{\partial(\rho r u)}{\partial x}+\frac{\partial(\rho r v)}{\partial r}+\frac{\partial \rho w}{\partial \omega}=0 \tag{2.9}$$

$$\begin{aligned}\rho\left(u\frac{\partial H}{\partial x}+v\frac{\partial H}{\partial r}+\frac{w}{r}\frac{\partial H}{\partial \omega}\right)&=\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\mu}{\sigma}\frac{\partial H}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \omega}\left(\frac{\mu}{\sigma}\frac{\partial H}{\partial \omega}\right)+\\&+\frac{1}{r}\frac{\partial}{\partial r}\left[\left(1-\frac{1}{\sigma}\right)r\mu\frac{\partial}{\partial r}\left(\frac{u^2}{2}\right)\right]+\frac{1}{r^2}\frac{\partial}{\partial \omega}\left[\left(1-\frac{1}{\sigma}\right)\mu\frac{\partial}{\partial \omega}\left(\frac{u^2}{2}\right)\right]\end{aligned} \tag{2.10}$$

$$p=\frac{\varkappa-1}{\varkappa}\rho h,\qquad \mu=\mu\left(\frac{h}{h_0}\right),\quad h=H-\frac{u^2}{2},\qquad h_0=\frac{1}{2} \tag{2.11}$$

For the case of axisymmetric flow, for which $w = 0$ and $\partial/\partial\omega = 0$, equation (2.6), in which the pressure is to be taken to be dependent only on x within the accuracy adopted, and equations (2.9), (2.10) and (2.11) form a closed system of boundary layer equations in which transverse curvature is taken into account. Once these equations are solved, the change of pressure p_1 across the boundary layer may be found from equation (2.7). Equation (2.8) is satisfied identically. In the absence of axial symmetry, equations (2.7) and (2.8) are solved together with the others.

On the body, whose equation is $r_0 = (\tau/\delta)R_b\ (x_0,\ \omega_0)$, the usual conditions of no-slip and equality of gas temperature and wall temperature are satisfied (or else the condition for heat flow into the wall)

$$u=0,\ v=0,\ H=h_b(x,\omega)\ (\text{ or } \partial H/\partial n=N_b(x,\omega)\ \text{ for } r=\frac{\tau}{\delta}R_b(x,w) \tag{2.12}$$

Here $h_b(x, \omega)$ is the enthalpy in terms of U_∞^2, corresponding to the given temperature distribution on the body surface (correspondingly, $N_b(x, \omega)$ is the given distribution of heat flux on the surface $R_b(x, \omega)$ and $\partial/\partial n$ is the derivative with respect to the normal to the body surface).

On the boundary between the viscous and inviscid flows, we have, taking into account the estimates (2.3), and to the accuracy adopted

$$u = 1, \quad v = R_\delta'(x), \quad w = 0, \quad H = {}^1/_2 \quad \text{for } r = R_b(x) \quad (2.13)$$

Inside the viscous flow region $h \sim 1$, while outside it $h \sim \delta^2$; therefor the error introduced by taking $h = 0$ at the boundary does not exceed that allowable. Condition $h = 0$ at the edge of the boundary layer was investigated in [2-4] for axisymmetric flows. Condition (2.13) for the radial velocity component v follows from the requirement that there be no flux across the surface $R_\delta(x)$.

Besides (2.13), we require that, in approaching the surface $r = R_\delta(x)$ from the viscous flow side, the viscous stresses and heat fluxes approach zero

$$\lim \mu \frac{\partial u}{\partial x} = \lim \mu \frac{\partial v}{\partial x} = \lim \mu \frac{\partial w}{\partial x} = \lim \mu \frac{\partial H}{\partial x} = 0$$

$$\lim \mu \frac{\partial u}{\partial r} = \lim \mu \frac{\partial v}{\partial r} = \lim \mu \frac{\partial w}{\partial r} = \lim \mu \frac{\partial H}{\partial r} = 0 \quad \text{for } r \to R_b \quad (2.14)$$

We write these in the form of limiting expressions because on the actual surface $r = R_\delta(x)$ the r-derivatives of the functions to be determined go to infinity (see Section 5).

The method of solution of the problem is as follows: assigning the function $r = R_\delta(x)$, and taking into account the tangency condition and the conditions at the shock wave (2.5), we solve the inviscid problem (2.4), thus determining the pressure p on the surface $r = R_\delta(x)$ as a functional of $R_\delta(x)$. (In the specific case where Newton's formula may be used, we have simply $p = (R_\delta')^2$.) Putting the value of p thus obtained in equation (2.6) and solving the system (2.6) to (2.11) with boundary conditions (2.12) to (2.14), we determine the viscous flow and find $R_\delta(x)$. In accordance with the similarity laws for hypersonic viscous flow [3,4], making use of a convenient form of writing them [8], we find that the solution depends on four dimensionless parameters, $K_1 = M_\infty\delta$, $K_2 = \tau^2/\delta^2 = \tau^2\sqrt{R_0}$, κ and σ, and two functions, $R_b(x, \omega)$ and $h_b(x, \omega)$ (or $N_b(x, \omega)$. For $M_\infty \to \infty$ the solution ceases to be dependent on K_1.

The system of equations (2.6) to (2.10) is of second order with respect to u, v, w and H and of first order with respect to p_1 (the density ρ is expressed in terms of H, in accordance with equation (2.11)).

In addition, the solution depends on one unknown function $R_\delta(x)$, through p and the boundary conditions. Thus, the solution of the resulting system of equations depends in all on 10 arbitrary functions. One of these functions, dependent on x, enters into the solution for p_1 as a parameter, since the quantity p_1 is differentiated with respect to r and ω in equations (2.7) and (2.8), and no boundary conditions are applied to p_1. This function, having the same order of magnitude as p_1, may, in principle, be determined from the solution in the following approximation. Its presence does not show up in any way in the solution for the remaining flow parameters – u, v, w, p, ρ and H. This function also does not influence the quantities determined from the solution for the drag force (due to the smallness of that force) and for the lift (due to the fact that it adds an axisymmetric increment to p_1). To solve the remaining nine arbitrary functions there are the boundary conditions (2.12), (2.13) and (2.14), of which there are just enough for a full solution of the problem, as will be shown in Section 5 for the example of a similarity solution.

3. To determine the order of magnitude of the forces acting on the body, we enclose the body in a control surface having the form of a cylinder with generators parallel to the velocity vector of undisturbed flow, $\mathbf{U}_\infty$ and by butt-ended planes perpendicular to $\mathbf{U}_\infty$. The rear butt-ended plane is separated from the nose of the body by a distance equal to the body length L. Instead of a cylindrical coordinate system, we introduce Cartesian coordinates Lx, Ly and Lz, where the x-axis is in the flow direction. Let $u_x U_\infty$, $u_y U_\infty$ and $u_z U_\infty$ be the corresponding velocity components, and the remaining notation as in Section 1.

For the velocity components, we have $u_y \sim \delta$, $u_z \sim \delta$ everywhere, $u_x = 1 + O(\delta^2)$ in the inviscid flow and $u_x \sim 1$ in the viscous flow.

Writing the momentum equation of a viscous gas in divergence form and applying Ostrogradski's formula to the volume of gas enclosed between the control surface and the body surface, we obtain for the forces X, Y and Z acting in the directions of the coordinate axes the expressions

$$\frac{X}{\rho_\infty U_\infty^2 L^2} = \iint\limits_S \left(1 - \rho u_x^2 + \frac{1}{\varkappa M_\infty^2} - p + \tau_{xx}\right) dS \tag{3.1}$$

$$\frac{Y}{\rho_\infty U_\infty^2 L^2} = \iint\limits_S (-\rho u_x u_y + \tau_{xy})\, dS, \qquad \frac{Z}{\rho_\infty U_\infty^2 L^2} = \iint\limits_S (-\rho u_x u_z + \tau_{xz})\, dS$$

Here, the integrals are taken over the area S of the rear butt-ended plane, between the body and the shock wave; τ_{xx} τ_{xy} and τ_{xz} are the components of the viscous stress tensor expressed in units of $\rho_\infty U_\infty^2$.

Each of the forces X, Y and Z represents the sum of pressure and friction forces. We shall denote by S_* and S_{**} the areas of the cross-section corresponding to inviscid and viscous flow, respectively (evidently $S = S_* + S_{**}$). Applying the equation of mass flux to the butt ended planes of the control surface, we have

$$S_* + S_{**} = \iint_{S_*} \rho u_x dS + \iint_{S_{**}} \rho u_x dS \tag{3.2}$$

Using the estimates made in Section 1, we have, with $S_* \sim S_{**} \sim \delta^2$

$$S_* + S_{**} = \iint_{S_*} \rho dS + o\left(\frac{\delta^4}{\varepsilon}\right) \tag{3.3}$$

Using equation (3.3), we estimate part of the integral in the expression for X

$$\iint_S (1 - \rho u_x^2)\, dS = S_* + S_{**} - \iint_{S_*} \rho u_x^2\, dS - \iint_{S_{**}} \rho u_x^2 dS = S_* + S_{**} - \\ - \iint_{S_*} \rho dS + o\left(\frac{\delta^4}{\varepsilon}\right) \sim \frac{\delta^4}{\varepsilon} \tag{3.4}$$

Estimating the remaining quantities, we obtain

$$\tau_{xx} \sim \frac{\mu_0 U_\infty}{L} \frac{1}{\rho_\infty U_\infty^2} = \frac{1}{R_0} \sim \frac{\delta^4}{\varepsilon}, \quad \frac{1}{M_\infty^2} \leqslant \delta^2, \quad p \sim \delta^2, \quad \frac{X}{\rho_\infty U_\infty^2 L^2} \sim \frac{\delta^4}{\varepsilon} \tag{3.5}$$

For evaluating the lift Y, we estimate the integral

$$\iint_S \rho u_x v_y dS = \iint_{S_*} \rho u_x v_y dS + \iint_{S_{**}} \rho u_x v_y dS \sim \frac{\delta^3}{\varepsilon}\delta^2 + \frac{\delta^2}{\varepsilon}\delta^3 \sim \frac{\delta^5}{\varepsilon} \tag{3.6}$$

The integral over S_* may be broken into two, over the half-ring $z > 0$ and over the half-ring $z < 0$. Although each of these integrals is of order δ^3, being of opposite sign their sum is of order δ^5/ε, due to the fact that the inviscid flow is axisymmetric up to error δ^2/ε. The integral over S_{**} is estimated in the usual way in terms of the magnitudes of the quantities in the integrand. Estimating τ_{xy}, we have, for Y and Z

$$\tau_{xy} \sim \frac{\mu_0 U_\infty}{L\delta} \frac{1}{\rho_\infty U_\infty^2} = \frac{1}{R_0 \delta} \sim \frac{\delta^3}{\varepsilon}, \quad \frac{Y}{\rho_\infty U_\infty^2 L^2} \sim \frac{Z}{\rho_\infty U_\infty^2 L^2} \sim \frac{\delta^5}{\varepsilon} \quad (\delta \geqslant \tau) \tag{3.7}$$

In the case of flow over a thin body of thickness τ at angle of attack $\alpha \sim \tau$, when the thickness of the viscous flow region is small compared to the body thickness, the pressure change in the circumferential direction changes its order of magnitude. Here we have, for Y and Z

$$\frac{Y}{\rho_\infty U_\infty{}^2L^2} \sim \frac{Z}{\rho_\infty U_\infty{}^2L^2} \sim \tau^3, \qquad [Y]_{\delta \gtrsim \tau} \sim \frac{\delta^2}{\varepsilon}\left(\frac{\delta}{\tau}\right)^3 [Y]_{\delta \leqslant \tau} \approx 0$$
$$[Z]_{\delta \gtrsim \tau} \sim \frac{\delta^2}{\varepsilon}\left(\frac{\delta}{\tau}\right)^3 [Z]_{\delta \leqslant \tau} \approx 0 \tag{3.8}$$

If we take $\delta \sim \tau$, then, with an error of order δ^2/ε, it is possible to neglect the left and side force in comparison to the corresponding quantities in the absence of viscous influence, i.e. to take $Y = Z = 0$.

If the estimate (3.8) is insufficient, and it is necessary to determine Y and Z for $\delta \gtrsim \tau$, this can be done after solving the problem formulated in Sections 1 and 2. In evaluating X, Y and Z, it is necessary to take into account not only the influence of the pressure forces (the lift and side force are produced by the increment of pressure Δp, dependent on r and ω and being of order δ^4/ε, in accordance with (1.8)), but also of the friction forces, since here those forces are of the same order.

4. In the case of flow over a slightly blunted cone (Ld is the diameter of the bluntness) a layer of low density gas is developed (the entropy layer). If this layer can be assumed to be viscous, then the above theory is valid without any changes. For the case of an inviscid entropy layer, assuming that its thickness νL satisfies the inequality $\nu \gtrsim \delta \gtrsim \tau$, we obtain from the equations of mass flux and entropy an estimate for ν

$$d^2 \sim \rho u_x \nu^2, \qquad \rho \sim \frac{p^{1/\varkappa}}{\varepsilon} \sim \frac{\nu^{2/\varkappa}}{\varepsilon} (p \sim \nu^2,\ u_x \sim 1), \qquad \nu \sim \varepsilon^{\frac{\varkappa}{2(\varkappa+1)}} d^{\frac{\varkappa}{\varkappa+1}} \tag{4.1}$$

In [5], where this estimate was obtained, ε was taken to be of order $\varepsilon \sim 1$. The change in pressure across the entropy layer [5] is $\Delta p \sim \nu^{2/\kappa}/\varepsilon$. For the boundary layer, $\Delta p \sim \delta^2/\varepsilon$. Thus, across the whole region occupied by the boundary and entropy layers, whose thickness is

$$\nu + \delta \sim \varepsilon^{\frac{\varkappa}{2(\varkappa+1)}} d^{\frac{\varkappa}{\varkappa+1}} + \left(\frac{\varepsilon}{R_0}\right)^{1/4} \tag{4.2}$$

the pressure is constant, with an error not exceeding $\nu^{2/\kappa}/\varepsilon$. Within the same error, the lift on the body at angle of attack $\alpha \lesssim \nu + \delta$ is equal to zero.

5. Let us assume that $M_\infty = \infty$, and the temperature of the surface of the body is constant or the body is insulated. For the case where the body surface (2.12) is given in the form $R_b(x, \omega) = \chi_b(\omega)x^{3/4}$, the equations derived in Section 2 permit a similarity solution, which is a generalization of the solution [6] for axisymmetric flow (for χ_b = const). We shall use the variables (2.1), dropping the subscript 0. We look for the equation of the outer boundary of the viscous flow region in the form $R_0(x) = \eta_\delta x^{3/4}$, where η_δ is a constant quantity which is to be determined. The equations (2.4) of inviscid axisymmetric flow with boundary conditions (2.5) have a similarity solution [9], for which the pressure p on the surface $R_\delta = \eta_\delta x^{3/4}$ is written

$$p = \Psi(\varkappa)\left(\frac{dR_\delta}{dx}\right)^2 = \frac{9\Psi(\varkappa)}{16}\frac{\eta_\delta^2}{\sqrt{x}} = \frac{c\eta_\delta^2}{\sqrt{x}} \tag{5.1}$$

From the calculations in [10], $\psi = 1.274$ and $c = 0.51$ for $\kappa = 1.4$. Equation (5.1) gives a functional relation between p and R_δ. We look for the remaining functions to be determined in equations (2.6) to (2.11) in the form

$$\begin{gathered} u = U(\eta,\omega), \quad v = x^{-1/4}V(\eta,\omega), \quad w = x^{-1/4}W(\eta,\omega) \\ \rho = x^{-1/2}R(\eta,\omega), \quad H = H(\eta,\omega), \quad p_1 = P_1/x, \quad \eta = r/x^{3/4} \end{gathered} \tag{5.2}$$

Equations (2.6) to (2.11) take the following form:

$$R\left(V - \frac{3}{4}U\eta\right)\frac{\partial U}{\partial\eta} + \frac{W}{\eta}\frac{\partial U}{\partial\omega} - \frac{c\eta_\delta^2}{2} = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\mu\eta\frac{\partial U}{\partial\eta}\right) + \frac{1}{\eta^2}\frac{\partial}{\partial\omega}\left(\mu\frac{\partial U}{\partial\omega}\right) \tag{5.3}$$

$$\begin{aligned} R\left(V - \frac{3}{4}U\eta\right)\frac{\partial V}{\partial\eta} + R\left(\frac{W}{\eta}\frac{\partial V}{\partial\omega} - \frac{UV}{4} - \frac{W^2}{\eta}\right) + \frac{\partial P_1}{\partial\eta} = -\frac{1}{4}\frac{\partial}{\partial\eta}\left(\mu\eta\frac{\partial U}{\partial\eta}\right) + \\ + \frac{4}{3}\frac{\partial}{\partial\eta}\left(\mu\frac{\partial V}{\partial\eta}\right) - \frac{2}{3}\frac{\partial}{\partial\eta}\left(\frac{\mu V}{\eta}\right) + \frac{1}{\eta^2}\frac{\partial}{\partial\omega}\left(\mu\frac{\partial V}{\partial\omega}\right) - \frac{2}{3}\frac{\partial}{\partial\eta}\left(\frac{\mu}{\eta}\frac{\partial W}{\partial\omega}\right) + \\ + \frac{\partial}{\partial\omega}\left[\mu\frac{\partial}{\partial\eta}\left(\frac{W}{\eta}\right)\right] + \frac{2\mu}{\eta}\frac{\partial}{\partial\eta}\left(\frac{V}{\eta}\right) - \frac{2\mu}{\eta^2}\frac{\partial W}{\partial\omega} \end{aligned} \tag{5.4}$$

$$\begin{aligned} R\left(V - \frac{3}{4}U\eta\right)\frac{\partial W}{\partial\eta} + R\left(\frac{W}{\eta}\frac{\partial W}{\partial\omega} - \frac{UW}{4} + \frac{VW}{\eta}\right) + \frac{1}{\eta}\frac{\partial P_1}{\partial\omega} = -\frac{3}{4}\frac{\partial}{\partial\eta}\left(\mu\frac{\partial U}{\partial\omega}\right) + \\ + \frac{1}{2}\frac{\partial}{\partial\omega}\left(\mu\frac{\partial U}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(\frac{\mu}{\eta}\frac{\partial V}{\partial\omega}\right) - \frac{2}{3\eta}\frac{\partial}{\partial\omega}\left(\mu\frac{\partial V}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left[\mu\eta\frac{\partial}{\partial\eta}\frac{W}{\eta}\right] + \\ + \frac{4}{3\eta^2}\frac{\partial}{\partial\omega}\left(\mu\frac{\partial W}{\partial\omega}\right) + \frac{4}{3\eta^2}\frac{\partial}{\partial\omega}(\mu V) + \frac{2\mu}{\eta^2}\frac{\partial V}{\partial\omega} + 2\mu\frac{\partial}{\partial\eta}\left(\frac{W}{\eta}\right) \end{aligned} \tag{5.5}$$

$$\frac{\partial}{\partial\eta}(RV\eta) + \frac{\partial(RW)}{\partial\omega} - \frac{3\eta^2}{4}\frac{\partial(RU)}{\partial\eta} - \frac{\eta RU}{2} = 0 \tag{5.6}$$

$$\begin{aligned} R\left(V - \frac{3}{4}U\eta\right)\frac{\partial H}{\partial\eta} + \frac{W}{\eta}\frac{\partial H}{\partial\eta} = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{\mu\eta}{\sigma}\frac{\partial H}{\partial\eta}\right) + \frac{1}{\eta^2}\frac{\partial}{\partial\omega}\left(\frac{\mu}{\sigma}\frac{\partial H}{\partial\omega}\right) + \\ + \frac{1}{\eta}\frac{\partial}{\partial\eta}\left[\left(1 - \frac{1}{\sigma}\right)\mu\eta\frac{\partial}{\partial\eta}\frac{U^2}{2}\right] + \frac{1}{\eta^2}\frac{\partial}{\partial\omega}\left[\left(1 - \frac{1}{\sigma}\right)\mu\frac{\partial}{\partial\omega}\frac{U^2}{2}\right] \end{aligned} \tag{5.7}$$

$$c\eta_\delta^2 = \frac{(\varkappa - 1)}{\varkappa}Rh \qquad \mu = \mu\left(\frac{h}{h_0}\right), \qquad h = H - \frac{U^2}{2}, \qquad h_0 = \frac{1}{2} \tag{5.8}$$

The boundary conditions on the body (2.12) and at the edge of the viscous layer (2.13) take, respectively, the form

$$U=0,\quad V=0,\quad W=0,\quad H=h_b\qquad \text{for } \eta=(\tau/\delta)\chi_b(\omega) \tag{5.9}$$

$$U=1,\quad V={}^3/_4\eta_\delta,\ W=0,\quad H={}^1/_2$$

$$\lim\mu\frac{\partial U}{\partial\eta}=\lim\mu\frac{\partial V}{\partial\eta}=\lim\mu\frac{\partial W}{\partial\eta}=\lim\mu\frac{\partial H}{\partial\eta}=0 \text{ for } \eta\to\eta_b \tag{5.10}$$

Let us investigate the asymptotic behavior of the solution in the vicinity of the outer boundary of the viscous flow region, restricting ourselves, for simplicity, to the case where the Prandtl number $\sigma = 1$ and the coefficient of viscosity depends linearly on the enthalpy, $\mu = 2H - U^2$. On the basis of (5.10), in the vicinity $\eta = \eta_\delta$ it is possible to introduce small quantities (indicated by the subscript *) which approach zero for $\eta_* \to 0$

$$\eta=\eta_\delta+\eta_*,\quad H={}^1/_2+H_*$$
$$U=1+U_*,\quad V={}^3/_4\eta_\delta+V_*,\quad W=W_* \tag{5.11}$$

Here, $R\to\infty$ for $\eta_*\to 0$. Let us assume (it will be verified later) that all quantities with subscript * are of the same order of magnitude $O(\eta_*)$, so that the derivatives of these quantities with respect to η_* are of order unity. Differentiation with respect to ω does not change the order of the quantity. From equation (5.8) it follows that $R\sim\eta_*^{-1}$; taking into account the linearity of $\mu(\eta)$, we obtain $\mu\sim\eta_*$. Putting expression (5.11) into equations (5.3) to (5.8) and neglecting quantities of order η_* in comparison with unity, we find that the system of equations separates into two groups, which can be solved successively. We introduce the notation

$$V_\times=R\left(V_*-\frac{3\eta_*}{4}-\frac{3\eta_\delta U_*}{4}\right) \tag{5.12}$$

We represent equations (5.3), (5.6), (5.7) and (5.8) in the form

$$V_\times\frac{\partial U_*}{\partial\eta_*}-\frac{c\eta_\delta^2}{2}=\frac{\partial}{\partial\eta_*}\left(\mu\frac{\partial U_*}{\partial\eta_*}\right),\qquad \frac{\partial V_\times}{\partial\eta_*}+R=0,\qquad V_\times\frac{\partial H_*}{\partial\eta_*}=\frac{\partial}{\partial\eta_*}\left(\mu\frac{\partial H_*}{\partial\eta_*}\right)$$
$$c\eta_\delta^2=R\,(H_*-U_*)\,(\varkappa-1)\,/\,\varkappa,\qquad \mu=2\,(H_*-W_*) \tag{5.13}$$

We go over from the variables η_* and ω to the variables ζ and ω, where the Dorodnitsyn variable ζ is given by

$$\zeta=\int_{\eta_{*1}}^{\eta_*}R\,d\eta_*,\qquad \eta_{*1}=\eta_{*1}(\omega) \tag{5.14}$$

Here, the integral is taken at a constant value of ω, and $\eta_{*1}(\omega)$ is an arbitrary function. Equations (5.13) do not contain derivatives with respect to ω and therefore integrate easily. From the second equation of (5.13) we obtain

$$\frac{\partial V_\times}{\partial \zeta} + 1 = 0, \qquad V_\times = -\zeta + \gamma(\omega) = -\zeta \tag{5.15}$$

The arbitrary function $\gamma(\omega)$ resulting from the integration may be taken equal to zero, taking note of (5.14), without any loss of generality; this results only in a change of the function $\eta_{*1}(\omega)$.

Taking into account equations (5.12) to (5.15), the third equation of (5.13) and its solution are written as

$$\lambda \frac{\partial^2 H_*}{\partial \zeta^2} + \zeta \frac{\partial H_*}{\partial \zeta} = 0 \qquad \left(\lambda = \frac{2\varkappa c \eta_\delta^2}{\varkappa - 1}\right) \tag{5.16}$$

$$H_* = f_1(\omega) \int_\zeta^\infty \exp\left(-\frac{\zeta^2}{2\lambda}\right) d\zeta \approx f_1(\omega) \frac{\exp(-\zeta^2/2\lambda)}{\zeta} \left(1 - \frac{\lambda}{\zeta^2} + \ldots\right)$$

We have required that $\eta_* \to 0$, and, consequently, $H_* \to 0$ for $\zeta \to \infty$. It is not difficult to show that, if ζ remains finite for $\eta_* \to 0$, then the condition of zero heat flux and viscous stress at the edge of the viscous flow region is violated. Thus, for $\eta_* \ll 1$, we have $\zeta \gg 1$, which makes it valid to use asymptotic expansions in (5.16) and below.

The first equation of (5.13) is written as

$$\lambda \frac{\partial^2 U_*}{\partial \zeta^2} + \zeta \frac{\partial U_*}{\partial \zeta} + \varepsilon (H_* - U_*) = 0 \qquad \left(\varepsilon = \frac{\varkappa - 1}{2\varkappa}\right) \tag{5.17}$$

where H_* is given by (5.16). The particular solution of equation (5.17) is evidently $U_* = H_*$. We look for a solution of the homogeneous equation, which depends on an arbitrary function $f_2(\omega)$, in the form of a generalized power series in powers of ζ^{-1}, multiplied by $\exp(-\zeta^2/2\lambda)$. The result is (noting that $U_* \to 0$ for $\zeta \to \infty$)

$$U_* = \frac{f_1(\omega)\exp(-\zeta^2/2\lambda)}{\zeta}\left(1 - \frac{\lambda}{\zeta^2} + \ldots\right) + \frac{f_2(\omega)\exp(-\zeta^2/2\lambda)}{\zeta^{1+\varepsilon}}\left[1 - \frac{\lambda(1+\varepsilon)(2+\varepsilon)}{2\zeta^2} + \ldots\right] \tag{5.18}$$

Using the fourth of equations (5.13), we find for the density

$$R = \frac{\lambda}{2(H_* - U_*)} = -\frac{\lambda \zeta^{1+\varepsilon} \exp(\zeta^2/2\lambda)}{2 f_2(\omega)}\left[1 + \frac{\lambda(1+\varepsilon)(2+\varepsilon)}{2\zeta^2} + \ldots\right] \tag{5.19}$$

From equation (5.14) we have

$$\eta_* = -\int_\zeta^\infty \frac{d\zeta}{R} = \frac{2 f_2(\omega)\exp(-\zeta^2/2\lambda)}{\zeta^{2+\varepsilon}}\left[1 + \frac{\lambda(2+\varepsilon)(1-\varepsilon)}{2\zeta^2} + \ldots\right] \tag{5.20}$$

Since ζ goes to infinity for $\eta_* \to 0$, the solution does not depend on the arbitrary function $\eta_{*1}(\omega)$ in equation (5.13).

From equations (5.12), (5.15), (5.18), (5.19) and (5.20), we have

$$V_* = \frac{3\eta_\delta f_1(\omega)}{4\zeta}\exp\left(-\frac{\zeta^2}{2\lambda}\right)\left(1-\frac{\lambda}{\zeta}+\ldots\right)+ \tag{5.21}$$
$$+\frac{2f_2(\omega)\exp(-\zeta^2/2\lambda)}{\lambda\zeta^\varepsilon}\left[1+\frac{3\lambda\eta_\delta}{8\zeta}-\frac{\lambda(\varepsilon^2+3\varepsilon+0.5)}{2\zeta^2}+\ldots\right]$$

All the functions of ω are periodic with period 2π. Equations (5.4) and (5.5), after dropping quantities of order η_* compared to unity, can be written in the new variables

$$V_\times\frac{\partial V_*}{\partial\zeta}-\frac{3\eta_\delta}{16}+\frac{\partial P_1}{\partial\zeta}=\frac{4\lambda}{3}\frac{\partial^2 V_*}{\partial\zeta^2}$$
$$RV_\times\frac{\partial W_*}{\partial\zeta}+\frac{RW_*}{2}+\frac{1}{\eta_\delta}\left(\frac{\partial P_1}{\partial\omega}\right)_{\zeta=\mathrm{const}}+\frac{1}{\eta_\delta}\frac{\partial P_1}{\partial\zeta}\left(\frac{\partial\zeta}{\partial\omega}\right)_{\eta=\mathrm{const}}=R\lambda\frac{\partial^2 W_*}{\partial\zeta^2} \tag{5.22}$$

The derivative of P_1 with respect to ω with η = const in equation (5.5) has been replaced in the second equation of (5.22) by the derivative with respect to ω with ζ = const.

Integrating the first equation of (5.22) with respect to ζ, taking (5.15) into account, we have

$$P_1=\frac{3\eta_\delta}{16}\zeta+\zeta V_*+\frac{4\lambda}{3}\frac{\partial V_*}{\partial\zeta}-\int V_* d\zeta+\int_0^\omega f_3(\omega)\,d\omega+C \tag{5.23}$$

The arbitrary function of ω which results from the integration in (5.23) can be conveniently written in the form of a sum of an integral of another arbitrary function $f_3(\omega)$ and a certain constant C. The condition of closure of the body contour, taking account of the periodicity of the functions $f_1(\omega)$ and $f_2(\omega)$, gives

$$P_1(\zeta,0)=P_1(\zeta,2\pi)\quad\text{or}\quad\int_0^{2\pi} f_3(\omega)\,d\omega=0 \tag{5.24}$$

We differentiate through equation (5.20) with respect to ω, taking η_* = const, finally obtaining

$$\left(\frac{\partial\zeta}{\partial\omega}\right)_\eta=\frac{\lambda}{\zeta f_2(\omega)}\frac{df_2}{d\omega}\left[1-\frac{\lambda(2+\varepsilon)}{\zeta^2}+\ldots\right] \tag{5.25}$$

Using equations (5.15), (5.19), (5.23) and (5.25) and neglecting terms of order η_* compared to unity, we transform the second equation of (5.22) to the form

$$\lambda\frac{\partial^2 W_*}{\partial\zeta^2}+\zeta\frac{\partial W_*}{\partial\zeta}-\frac{W_*}{2}=-\frac{3}{8\zeta^{2+\varepsilon}}\frac{df_2}{\partial\omega}\exp\left(-\frac{\zeta^2}{2\lambda}\right)\times$$

$$\times\left[1-\frac{\lambda(2+\varepsilon)(3+\varepsilon)}{2\zeta^2}+\ldots\right]-\frac{2f_3(\omega)f_2(\omega)}{\lambda\eta_\delta\zeta^{1+\varepsilon}}\left[1-\frac{\lambda(1+\varepsilon)(2+\varepsilon)}{2\zeta^2}+\ldots\right] \quad (5.26)$$

The solution of this equation can be written as

$$W_* = -\frac{3}{4(1+2\varepsilon)}\frac{df_2}{d\omega}\frac{\exp(-\zeta^2/2\lambda)}{\zeta^{2+\varepsilon}}\left[1-\frac{\lambda(2+\varepsilon)(3+\varepsilon)}{2\zeta^2}+\ldots\right]+$$
$$+\frac{4f_3(\omega)f_2(\omega)\exp(-\zeta^2/2\lambda)}{\lambda\eta_\delta(1-2\varepsilon)\zeta^{1+\varepsilon}}\left[1-\frac{\lambda(1+\varepsilon)(2+\varepsilon)(1-2\varepsilon)(3-2\varepsilon)}{2(1+2\varepsilon)(3+2\varepsilon)\zeta^2}+\ldots\right]+$$
$$+\frac{f_4(\omega)}{\zeta^{3/2}}\exp\left(-\frac{\zeta^2}{2\lambda}\right)\left(1+\frac{15\lambda}{8\zeta^2}+\ldots\right) \quad (5.27)$$

The asymptotic solution, (5.16), (5.18), (5.19), (5.20), (5.21), (5.23) and (5.27), of the original equations depends on four functions, $f_1(\omega)$, $f_2(\omega)$, $f_3(\omega)$ and $f_4(\omega)$ and one constant quantity, η_δ (the constant C in equation (5.23) for P_1 evidently has no effect on the result. In principle, this quantity can be determined from a higher order approximation of the solution. In other words, the given solution makes it possible to find P_1 correctly up to an additive constant). It is essential that $f_3(\omega)$ satisfy condition (5.24). The dependence of the solution on one constant quantity, η_δ, and a periodic function $f_3(\omega)$, whose integral over a period is equal to zero, is equivalent to the dependence on a certain arbitrary function on which no conditions are imposed. The remaining functions $f_1(\omega)$, $f_2(\omega)$ and $f_4(\omega)$ will be arbitrary. Thus, the asymptotic solution depends on four arbitrary functions. On the body there are just this many boundary conditions (5.9).

As follows from the solution, all the functions sought, except the density, behave near $\eta = \eta_\delta$ like $\zeta^\alpha \exp(-\zeta^2/2\lambda)$ + const, which with (5.20) can be rewritten as $(\eta_\delta - \eta)[\ln(\eta_\delta - \eta)]^\alpha$ + const, where α is some number. Thus, for $\eta = \eta_\delta$, the derivatives of the quantities to be determined have logarithmic singularities. The density $R \sim (\eta_\delta - \eta)^{-1} \ln(\eta_\delta - \eta)$ goes to infinity for $\eta = \eta_\delta$.

Once the asymptotic solution is obtained, it can be refined. The condition for which all quantities with subscript * are of order η_* means that these quantities, including η_*, may be represented in the form of a product of $\exp(-\zeta^2/2\lambda)$ with a certain generalized power series in powers of ζ^{-1}.

In obtaining the asymptotic solution, the quantity η_*^2 was neglected in comparison with quantities of order η_*. If the omitted terms of order $\eta_*^2 \sim \exp(-\zeta^2/2\lambda)$, determined from the first approximation, are now placed in the right-hand side of each of the simplified equations, for example, equation (5.17), then it is possible to obtain a correction to the solution which is proportional to the product of $\exp(-\zeta^2/2\lambda)$ with a certain generalized power series in powers of ζ^{-1}.

Repeating this procedure, we find that the asymptotic solution for U_* has the following structure:

$$U_* = \sum_{k=0,\ n=0}^{\infty} \frac{a_{nk}(\omega)}{\zeta^{n+\alpha}} \exp\left[\left(-\frac{\zeta^2}{2\lambda}\right)(1+k)\right], \qquad \alpha = \alpha(k) \tag{5.28}$$

In fact, the asymptotic expansions for the remaining quantities with subscript * have the same form. The expansion for R begins with $k = -2$. In equation (5.28) the coefficients $\alpha_{nk}(\omega)$ of the double series depend on the functions $f_1(\omega)$, $f_2(\omega)$, $f_3(\omega)$, $f_4(\omega)$ and the constant η_δ, as may be easily shown.

6. We investigate the case of flow over a thin body of revolution at small angle of attack $\alpha \ll \tau \leqslant \delta$. With this, it is possible to linearize around the solution of the corresponding axisymmetric flow. Instead of equations (2.1), we may write for the functions to be determined

$$\begin{gathered} x = x_0, \qquad r = \delta r_0, \qquad \omega = \omega_0, \qquad u = u_0(x_0, r_0) + u_\alpha(x_0, r_0, \omega_0) \\ v = \delta v_0(x_0, r_0) + v_\alpha(x_0, r_0, \omega_0), \qquad w = w_\alpha(x_0, r_0, \omega_0) \\ p = \delta^2 p_0(x_0) + \delta^4 p_1(x_0, r_0) + p_\alpha(x_0, r_0, \omega_0) \\ \rho = \delta^2 \rho_0(x_0, r_0) + \rho_\alpha(x_0, r_0, \omega_0), \qquad H = \delta^2 H_0(x_0, r_0) + H_\alpha(x_0, r_0, \omega_0) \end{gathered} \tag{6.1}$$

Quantities with subscript 0 and the quantity p_1 in the expression for the pressure, being of order unity, correspond to the axisymmetric flow. It is assumed that the angle α is so small that the perturbations with subscript α, which depend on the angle of attack, are much smaller than the basic quantities. The equation of the surface of the body, whose axis of symmetry makes a small angle α with the x-axis of a cylindrical coordinate system, is written in the form

$$r = \tau R_b(x) + R_\alpha(x, \omega) \quad \text{or} \quad r_0 = \frac{\tau}{\delta} R_b(x_0) - \frac{\alpha}{\delta} x_0 \cos\omega_0 \tag{6.2}$$

We reduce the boundary conditions (2.12) on the surface (6.2) to the body surface for $\alpha = 0$. Taking into account that quantities with subscript 0 satisfy those boundary conditions on the body surface for $\alpha = 0$, we have, for example, for u (neglecting products of small perturbations with subscript α)

$$u = u_0 + u_\alpha \approx [u_0 + u_\alpha] + \frac{R_\alpha}{\delta}\left[\frac{\partial}{\partial r_0}(u_0 + u_\alpha)\right] \approx [u_\alpha] + \frac{R_\alpha}{\delta}\left[\frac{\partial u_0}{\partial r_0}\right] = 0 \tag{6.3}$$

where the square brackets indicate that the given quantity is taken on the body surface at $\alpha = 0$.

The remaining boundary conditions are written similarly (the condition for the enthalpy is taken in the form $H = h_b$). Taking (6.2) into account, we obtain

$$u_\alpha = \frac{\alpha}{\delta} x_0 \frac{\partial u_0}{\partial r_0} \cos\omega_0, \qquad v_\alpha = \alpha x_0 \frac{\partial v_0}{\partial r_0} \cos\omega_0, \qquad w_\alpha = 0$$
$$H_\alpha = \frac{\alpha}{\delta} x_0 \frac{\partial H_0}{\partial r_0} \cos\omega_0 \qquad \text{for } r_0 = \frac{\tau}{\delta} R_b(x_0) \tag{6.4}$$

Equations (6.4) determine the orders of the quantities u_α, v_α and H_α, and equations (2.6) to (2.11) the orders of the remaining quantities with subscript α. In addition, as a result of the linearization and on account of (6.4), it is possible to find the dependence on ω_0; perturbations depending on angle of attack effects take the form

$$u_\alpha = \frac{\alpha}{\delta} u' \cos\omega_0, \qquad p_\alpha = \alpha\delta^3 p' \cos\omega_0, \qquad H_\alpha = \frac{\alpha}{\delta} H' \cos\omega$$
$$v_\alpha = \alpha v' \cos\omega_0, \qquad \rho_\alpha = \alpha\delta\rho' \cos\omega_0, \qquad w_\alpha = \alpha w' \sin\omega_0 \tag{6.5}$$

where primed quantities are of order unity. In linearizing equations (2.6) to (2.11) an error of order α/δ is introduced, as may be easily seen. Equation (2.2) of the outer boundary of the viscous flow region takes the form

$$r_0 = R_\delta(x_0) + \alpha\delta R'(x_0) \cos\omega \approx R_\delta(x_0) \tag{6.6}$$

Boundary conditions (2.13) on this surface assume a simple form, with an error $\alpha\delta$ which is significantly smaller than the allowable error α/δ

$$u' = 0, \qquad v' = 0, \qquad w' = 0, \qquad H' = 0 \qquad \text{for } r_0 = R_\delta(x_0) \tag{6.7}$$

The problem is solved in the following manner: first, u_0, v_0, w_0, p_0, ρ_0 and H_0 (it is not necessary to determine p_1) are found from the usual equations for the axisymmetric boundary layer, after which the perturbation quantities are found from the linearized equations (2.6) to (2.11) together with the boundary conditions (6.4) and (6.7).

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